LARGE DIMENSIONAL HOMOMORPHISM SPACES BETWEEN WEYL MODULES AND SPECHT MODULES

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Abstract. We give a family of pairs of Weyl modules for which the corresponding homomorphism space is at least 2-dimensional. Using this result we show that for fixed parameters $e > 0$ and $p \geq 0$ there exist arbitrarily large homomorphism spaces between pairs of Weyl modules.

1. Introduction

Let F be a field of characteristic $p \geq 0$. Take $q \in F^\times$ with the property that $1+q+\ldots+q^{f-1} = 0 \in F$ for some integer $2 \leq f < \infty$ and let $e \geq 2$ be minimal with this property. For $n \geq 0$, we write $\mathcal{H}_n = \mathcal{H}_{F,q}(\mathfrak{S}_n)$ to denote the Hecke algebra of the symmetric group $\mathfrak{S}_n$ and $\mathcal{S}_n = \mathcal{S}_{F,q}(\mathfrak{S}_n)$ to denote the corresponding q-Schur algebra. For each partition μ of n, we may define a $\mathcal{H}_n$-module S^μ, known as a Specht module, and an $\mathcal{S}_n$-module $\Delta(\mu)$, known as a Weyl module. Recall that if μ and λ are partitions of n then

$$\dim(\mathrm{Hom}_{\mathcal{H}_n}(S^\mu, S^\lambda)) \geq \dim(\mathrm{Hom}_{\mathcal{S}_n}(\Delta(\mu), \Delta(\lambda))$$

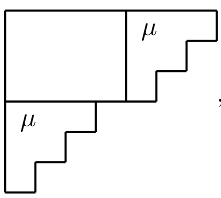
with equality if $q \neq -1$ [2]. Despite much investigation, there are few known examples of Weyl modules $\Delta(\mu)$ and $\Delta(\lambda)$ such that $\dim(\mathrm{Hom}_{\mathcal{S}_n}(\Delta(\mu), \Delta(\lambda))) > 1$. The first such pairs were recently exhibited by Dodge [3]. Working in the symmetric group algebra and using results of Chuang and Tan [1] on the radical filtrations of Specht modules belonging to Rouquier blocks, he showed that for any k satisfying $k(k+1)/2+1 < p$ there exist partitions μ and λ of some integer n such that $\dim(\mathrm{Hom}_{F\mathfrak{S}_n}(S^\mu, S^\lambda)) = k$. In particular, for $p \geq 5$ there exist Specht modules, and hence Weyl modules, such that the corresponding homomorphism space is at least 2-dimensional. Using Lemma 1.1 below, Dodge's result proves the following: Let F be a field of characteristic $p \geq 5$. Then given any integer $l \geq 0$ there exist partitions α and β of some integer m such that $\dim(\mathrm{Hom}_{F\mathfrak{S}_m}(S^\alpha, S^\beta)) \geq l$.

Lemma 1.1. *Suppose μ and λ are partitions of an integer n such that $\dim(\mathrm{Hom}_{\mathcal{S}_n}(\Delta(\mu), \Delta(\lambda))) = k$. Then there exist partitions α and β of some integer m such that $\dim(\mathrm{Hom}_{\mathcal{S}_m}(\Delta(\alpha), \Delta(\beta))) = k^2$.*

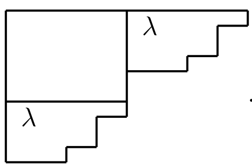
Proof. We may assume $k \geq 1$. If $\mu = (\mu_1, \ldots, \mu_a)$ and $\lambda = (\lambda_1, \ldots, \lambda_b)$ then, since $\mathrm{Hom}_{\mathcal{S}_n}(\Delta(\mu), \Delta(\lambda)) \neq \{0\}$, we have $\lambda \trianglerighteq \mu$ so that $a \geq b$ and $\lambda_1 \geq \mu_1$. Define partitions α and β by

$$\alpha_i = \begin{cases} \mu_i + \lambda_1, & 1 \leq i \leq a, \\ \mu_{i-a}, & a+1 \leq i \leq 2a, \end{cases} \qquad \beta_i = \begin{cases} \lambda_i + \lambda_1, & 1 \leq i \leq a, \\ \lambda_{i-b}, & a+1 \leq i \leq 2a, \end{cases}$$

so that

$\alpha =$ 

,

$\beta =$ 

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Then $\dim(\mathrm{Hom}_{\mathcal{S}_n}(\Delta(\alpha), \Delta(\beta))) = k^2$ by the generalized row and column removal theorems [6, Theorem 3.1] or [4, Prop. 10.4]. $\square$

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In this paper, we exhibit pairs of partitions such that the homomorphism space between the corresponding Weyl modules is at least 2-dimensional. In fact, we believe that it is exactly 2-dimensional, but this would be considerably harder to prove.

Theorem 1.2. *For $a \geq b \geq c+1 \geq 4$, define partitions*

$$\mu = \mu(a,b,c,e) = (ae-3, be-3, ce-3, e-1, e-1),$$
$$\lambda = \lambda(a,b,c,e) = ((a+2)e-5, be-3, ce-3)),$$

of some integer n. Then $\dim(\mathrm{Hom}_{\mathcal{S}_n}(\Delta(\mu), \Delta(\lambda))) \geq 2$.

Using Lemma 1.1, this is sufficient to prove the following result.

Theorem 1.3. *Given any integer $l \geq 0$ there exist partitions α and β of some integer m such that*

$$\dim(\mathrm{Hom}_{\mathcal{S}_m}(\Delta(\alpha), \Delta(\beta))) \geq l; \text{ and hence}$$
$$\dim(\mathrm{Hom}_{\mathcal{H}_m}(S^\alpha, S^\beta)) \geq l.$$

If the results of Chaung and Tan [1] hold for the q-Schur algebra (rather than just the Schur algebra) then the proof of Theorem 1.3 almost follows from the work of Dodge (and Lemma 1.1): only the cases $e = 2, 3, 4$ would not be covered. We note that Lemma 1.1 is the only result we know that allows us to build large homomorphism spaces from smaller ones; for example, for small e we do not know of any pair of partitions such that the homomorphism space between the corresponding Weyl modules has dimension 3.

2. Proof of Theorem 1.2

In this section, we give the proof of the main result. Fix a field F and an element $q \in F^\times$ such that $e = \min\{f \geq 2 \mid 1 + q + \ldots + q^{f-1} = 0\}$ exists. For $n \geq 0$ let $\mathcal{S}_n = \mathcal{S}_{F,q}(\mathfrak{S}_n)$ and $\mathcal{H}_n = \mathcal{H}_{F,q}(\mathfrak{S}_n)$. The characteristic of the field plays no further role in this paper. We first recall a method to determine the dimension of the homomorphism space between a pair of Weyl modules. For full details, we refer the reader to [5, Section 2.2].

2.1. **Homomorphism spaces.** Fix partitions λ and μ of n. For every composition ν of n, we define $m_\nu \in \mathcal{H}_n$ and a cyclic right $\mathcal{H}_n$-module $M^\nu = m_\nu \mathcal{H}$. Let $\mathcal{T}_{\mathrm{r}}(\lambda, \nu)$ denote the set of row-standard λ-tableaux of type ν, with $\mathcal{T}_0(\lambda, \nu) \subseteq \mathcal{T}_{\mathrm{r}}(\lambda, \nu)$ the subset of semistandard tableaux. For each $\mathsf{T} \in \mathcal{T}_{\mathrm{r}}(\lambda, \nu)$ we define a $\mathcal{H}_n$-homomorphism $\Theta_{\mathsf{T}} : M^\nu \to S^\lambda$ such that $\{\Theta_{\mathsf{T}} \mid \mathsf{T} \in \mathcal{T}_0(\lambda, \nu)\}$ are linearly independent.

Let $\ell(\nu)$ denote the number of parts of any composition ν. For $1 \leq d < \ell(\mu)$ and $1 \leq t \leq \mu_{d+1}$ we define an element $h_{d,t} \in \mathcal{H}_n$. Let $\mathrm{EHom}_{\mathcal{H}_n}(M^\mu, S^\lambda)$ be the space spanned by $\{\Theta_{\mathsf{T}} \mid \mathsf{T} \in \mathcal{T}_0(\lambda, \mu)\}$ and let

$$\Psi(\mu, \lambda) = \{\Theta \in \mathrm{EHom}_{\mathcal{H}_n}(M^\mu, S^\lambda) \mid \Theta(m_\mu h_{d,t}) = 0 \text{ for all } 1 \leq d < \ell(\mu), 1 \leq t \leq \mu_{d+1}\}.$$

This definition is motivated by the following result which follows from [5, Theorem 2.2] and the remark following [5, Corollary 2.4].

Lemma 2.1.

$$\Psi(\mu, \lambda) \cong_F \mathrm{Hom}_{\mathcal{S}_n}(\Delta(\mu), \Delta(\lambda)).$$

We therefore want to determine $\Psi(\mu, \lambda)$. First we set up some notation. If $\mathsf{T} \in \mathcal{T}_{\mathrm{r}}(\lambda, \nu)$, let T^i_j denote the number of entries of T which lie in row j and which are equal to i. We extend this definition by setting $\mathsf{T}^{>i}_j = \sum_{k>i} \mathsf{T}^k_j$, and similarly for other definitions. If $m \geq 0$ define

$$[m] = 1 + q + \ldots + q^{m-1} \in F.$$

Let $[0]! = 1$ and for $m \geq 1$, set $[m]! = [m][m-1]!$. If $m \geq j \geq 0$, set

$$\begin{bmatrix} m \\ j \end{bmatrix} = \frac{[m]!}{[j]![m-j]!}.$$

For integers m and j, if any of the conditions $m \geq j \geq 0$ fail we define $\begin{bmatrix} m \\ j \end{bmatrix} = 0$. Using Lemma 2.4 below or otherwise, it is straightforward to show that $\begin{bmatrix} m \\ k \end{bmatrix}$ is then well-defined for any $m, k \in \mathbb{Z}$, and may be considered as an element of $\mathbb{Z}[q]$.

Lemma 2.2 ([5] Proposition 2.7)**.** *Suppose that $\mathsf{T} \in \mathcal{T}_{\mathrm{r}}(\lambda,\mu)$. Choose d with $1 \le d < \ell(\mu)$ and t with $1 \le t \le \mu_{d+1}$. Let $\mathcal{S}$ be the set of row-standard tableaux obtained by replacing t of the entries in T which are equal to $d+1$ with d. Each tableau $\mathsf{S} \in \mathcal{S}$ will be of type $\nu(d,t)$ where*

$$\nu(d,t)_j = \begin{cases} \mu_j + t, & j = d, \\ \mu_j - t, & j = d+1, \\ \mu_j, & \textit{otherwise.} \end{cases}$$

Recall that $\Theta_{\mathsf{T}} : M^\mu \to S^\lambda$ and $\Theta_{\mathsf{S}} : M^{\nu(d,t)} \to S^\lambda$. Then

$$\Theta_{\mathsf{T}}(m_\mu h_{d,t}) = \sum_{\mathsf{S}\in\mathcal{S}} \left(\prod_{j=1}^{\ell(\lambda)} q^{\mathsf{T}^d_{>j}(\mathsf{S}^d_j - \mathsf{T}^d_j)} \begin{bmatrix} \mathsf{S}^d_j \\ \mathsf{T}^d_j \end{bmatrix} \right) \Theta_{\mathsf{S}}(m_{\nu(d,t)}).$$

Lemma 2.3 ([5] Proposition 2.9)**.** *Suppose λ is a partition of n and ν is a composition of n. Let $\mathsf{S} \in \mathcal{T}_{\mathrm{r}}(\lambda,\nu)$. Suppose $1 \le r \le \ell(\lambda)-1$ and that $1 \le d \le \ell(\nu)$. Let*

$$\mathcal{G} = \left\{ g = (g_1, g_2, \dots, g_{\ell(\nu)}) \mid g_d = 0, \sum_{i=1}^{\ell(\nu)} g_i = \mathsf{S}^d_{r+1} \textit{ and } g_i \le \mathsf{S}^i_r \textit{ for } 1 \le i \le \ell(\nu) \right\}.$$

For $g \in \mathcal{G}$, let $\bar{g}_{d-1} = \sum_{i=1}^{d-1} g_i$ and let U_g be the row-standard tableau formed from S by moving all entries equal to d from row $r+1$ to row r and for $i \ne d$ moving g_i entries equal to i from row r to row $r+1$. Then

$$\Theta_{\mathsf{S}} = (-1)^{\mathsf{S}^d_{r+1}} q^{-\binom{\mathsf{S}^d_{r+1}+1}{2}} q^{-\mathsf{S}^d_{r+1} S^{<d}_{r+1}} \sum_{g\in\mathcal{G}} q^{\bar{g}_{d-1}} \prod_{i=1}^{\ell(\nu)} q^{g_i \mathsf{S}^{<i}_{r+1}} \begin{bmatrix} \mathsf{S}^i_{r+1} + g_i \\ g_i \end{bmatrix} \Theta_{\mathsf{U}_g}.$$

In the following section, we apply these two lemmas to find elements of $\Psi(\mu,\lambda)$.

Example. Let $e = 2$. Take $\lambda = (7,5,3)$ and $\mu = (5,5,3,1,1)$. We identify a λ-tableau T of type $\nu \trianglerighteq \mu$ with the image $\Theta_{\mathsf{T}}(m_\nu) \in S^\lambda$. Recall that if $\lambda \not\trianglerighteq \nu$ then $\mathcal{T}_0(\lambda,\nu) = \emptyset$ so that we immediately have $\Theta(m_\mu h_{1,t}) = 0$ for $t = 3,4,5$ and $\Theta(m_\mu h_{2,3}) = 0$.

(1) Let $\Theta(m_\mu) = \ytableaushort{1111123,22223,345}$. Then

$$\begin{aligned}
\Theta(m_\mu h_{4,1}) &= [2]\,\ytableaushort{1111123,22223,344} \\
&= 0, \\
\Theta(m_\mu h_{3,1}) &= [2]\,\ytableaushort{1111123,22223,335} \\
&= 0, \\
\Theta(m_\mu h_{2,1}) &= q^4[2]\,\ytableaushort{1111122,22223,345} + [5]\,\ytableaushort{1111123,22222,345} + \ytableaushort{1111123,22223,245} \\
&= q^4[2]\,\ytableaushort{1111122,22223,345} + [5]\,\ytableaushort{1111123,22222,345} - \ytableaushort{1111123,22222,345} \\
&= 0, \\
\Theta(m_\mu h_{2,2}) &= q^4[2][5]\,\ytableaushort{1111122,22222,345} + q^4[2]\,\ytableaushort{1111122,22223,245} + [5]\,\ytableaushort{1111123,22222,245} \\
&= q^4[2][5]\,\ytableaushort{1111122,22222,345} - q^4[2]\,\ytableaushort{1111122,22222,345} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\Theta(m_\mu h_{1,1}) &= [6]\begin{smallmatrix}1&1&1&1&1&1&3\\2&2&2&2&3\\3&4&5\end{smallmatrix} + \begin{smallmatrix}1&1&1&1&1&2&3\\1&2&2&2&3\\3&4&5\end{smallmatrix}\\
&= [6]\begin{smallmatrix}1&1&1&1&1&1&3\\2&2&2&2&3\\3&4&5\end{smallmatrix} - [4]\begin{smallmatrix}1&1&1&1&1&1&3\\2&2&2&2&3\\3&4&5\end{smallmatrix} - q^3[2]\begin{smallmatrix}1&1&1&1&1&1&2\\2&2&2&3&3\\3&4&5\end{smallmatrix}\\
&= 0,\\
\Theta(m_\mu h_{1,2}) &= [6]\begin{smallmatrix}1&1&1&1&1&1&3\\1&2&2&2&3\\3&4&5\end{smallmatrix} + \begin{smallmatrix}1&1&1&1&1&2&3\\1&1&2&2&3\\3&4&5\end{smallmatrix}\\
&= -q^3[6][2]\begin{smallmatrix}1&1&1&1&1&1&1\\2&2&2&3&3\\3&4&5\end{smallmatrix} + q^3[3][2]\begin{smallmatrix}1&1&1&1&1&1&1\\2&2&2&3&3\\3&4&5\end{smallmatrix}\\
&= 0,
\end{aligned}$$

so that $\Theta \in \Psi(\mu,\lambda)$.

(2) Let

$$\Phi = \begin{smallmatrix}1&1&1&1&1&2&5\\2&2&2&2&4\\3&3&3\end{smallmatrix} + \begin{smallmatrix}1&1&1&1&1&2&4\\2&2&2&2&5\\3&3&3\end{smallmatrix} + \begin{smallmatrix}1&1&1&1&1&2&5\\2&2&2&2&3\\3&3&4\end{smallmatrix} + \begin{smallmatrix}1&1&1&1&1&2&4\\2&2&2&2&3\\3&3&5\end{smallmatrix} + \begin{smallmatrix}1&1&1&1&1&2&3\\2&2&2&2&5\\3&3&4\end{smallmatrix} + \begin{smallmatrix}1&1&1&1&1&2&3\\2&2&2&2&4\\3&3&5\end{smallmatrix}.$$

Then $\Phi \in \Psi(\mu,\lambda)$.

2.2. **Gaussian Polynomials.** In order to tell if a homomorphism Θ lies in $\Psi(\mu,\lambda)$ we record some results about the Gaussian polynomials $\begin{bmatrix} m \\ j \end{bmatrix}$. The first is well-known.

Lemma 2.4. *Suppose $m, j \geq 0$. Then*

$$\begin{aligned}
\begin{bmatrix} m+1 \\ j \end{bmatrix} &= \begin{bmatrix} m \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} m \\ j \end{bmatrix}\\
&= \begin{bmatrix} m \\ j \end{bmatrix} + q^{m-j+1} \begin{bmatrix} m \\ j-1 \end{bmatrix}.
\end{aligned}$$

Lemma 2.5 ([5] Lemma 2.6)**.** *Suppose $m, k \geq l \geq 0$. Then,*

$$\sum_{j \geq 0} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} l \\ j \end{bmatrix} \begin{bmatrix} m-j \\ k \end{bmatrix} = q^{l(m-k)} \begin{bmatrix} m-l \\ k-l \end{bmatrix}.$$

Lemma 2.6. *Suppose that $m \geq 0$ and write $m = m^*e + m'$ where $0 \leq m' < e$. If $m' < j \leq e-1$ then*

$$\begin{bmatrix} m \\ j \end{bmatrix} = 0.$$

Proof. Write

$$\begin{bmatrix} m \\ j \end{bmatrix} = \frac{[m][m-1]\ldots[m-j+1]}{[j][j-1]\ldots[1]}$$

so that one of the terms in the numerator and none of the terms in the denominator are equal to zero. □

The next lemma follows immediately.

Lemma 2.7. *Suppose $1 \leq j \leq e-1$. Then*

$$\begin{bmatrix} ae-1+j \\ j \end{bmatrix} = 0$$

for all $a \geq 0$.

Lemma 2.8. *Suppose $m \geq l \geq 0$, that $k \geq 1$ and that $a_1, \ldots, a_k \geq 0$ are such that $\sum_{i=1}^k a_i = m$. Then*

$$\sum_{c_1+\ldots+c_k=l} \prod_{i=1}^k q^{(a_i-c_i)(c_{i+1}+\ldots+c_k)} \begin{bmatrix} a_i \\ c_i \end{bmatrix} = \begin{bmatrix} m \\ l \end{bmatrix}.$$

Proof. The result is true for $m = 0$ so suppose that $m \geq 1$ and that the lemma holds for $m-1$. Using Lemma 2.4 and the inductive hypothesis,

$$\begin{aligned}
\sum_{c_1+\ldots+c_k=l} \prod_{i=1}^{k} q^{(a_i-c_i)(c_{i+1}+\ldots+c_k)} \begin{bmatrix} a_i \\ c_i \end{bmatrix} \\
&= \sum_{c_1+\ldots+c_k=l} \left(\prod_{i=1}^{k-1} q^{(a_i-c_i)(c_{i+1}+\ldots+c_k)} \begin{bmatrix} a_i \\ c_i \end{bmatrix} \right) \left(\begin{bmatrix} a_k-1 \\ c_k \end{bmatrix} + q^{a_k-c_k} \begin{bmatrix} a_k-1 \\ c_k-1 \end{bmatrix} \right) \\
&= \sum_{c_1+\ldots+c_k=l} \left(\prod_{i=1}^{k-1} q^{(a_i-c_i)(c_{i+1}+\ldots+c_k)} \begin{bmatrix} a_i \\ c_i \end{bmatrix} \right) \begin{bmatrix} a_k-1 \\ c_k \end{bmatrix} \\
&\quad + q^{a_1+\ldots+a_k-c_1-\ldots-c_k} \sum_{c_1+\ldots+c_k=l-1} \left(\prod_{i=1}^{k-1} q^{(a_i-c_i)(c_{i+1}+\ldots+c_k)} \begin{bmatrix} a_i \\ c_i \end{bmatrix} \right) \begin{bmatrix} a_k-1 \\ c_k \end{bmatrix} \\
&= \begin{bmatrix} m-1 \\ l \end{bmatrix} + q^{m-l} \begin{bmatrix} m-1 \\ l-1 \end{bmatrix} \\
&= \begin{bmatrix} m \\ l \end{bmatrix}
\end{aligned}$$

as required.

An alternative proof may be constructed by counting the number of l-dimensional vector spaces of an m-dimensional vector space over the finite fields. $\square$

2.3. **Elements of $\Psi(\mu, \lambda)$.** We are now ready to prove Theorem 1.2. Fix $a \geq b \geq c+1 \geq 4$ and define partitions

$$\begin{aligned}
\mu &= \mu(a,b,c,e) = (ae-3, be-3, ce-3, e-1, e-1), \\
\lambda &= \lambda(a,b,c,e) = ((a+2)e-5, be-3, ce-3)),
\end{aligned}$$

of some integer n. If $\mathsf{T} \in \mathcal{T}_{\mathrm{r}}(\lambda, \nu)$ for some $\nu \trianglerighteq \mu$, recall that T^i_j is the number of entries equal to i in row j of T. We denote T by

$$\mathsf{T} = \begin{array}{l} 1^{\mathsf{T}^1_1}\, 2^{\mathsf{T}^2_1}\, 3^{\mathsf{T}^3_1}\, 4^{\mathsf{T}^4_1}\, 5^{\mathsf{T}^5_1} \\ 1^{\mathsf{T}^1_2}\, 2^{\mathsf{T}^2_2}\, 3^{\mathsf{T}^3_2}\, 4^{\mathsf{T}^4_2}\, 5^{\mathsf{T}^5_2} \\ 1^{\mathsf{T}^1_3}\, 2^{\mathsf{T}^2_3}\, 3^{\mathsf{T}^3_3}\, 4^{\mathsf{T}^4_3}\, 5^{\mathsf{T}^5_3} \end{array},$$

where we omit terms if $\mathsf{T}^i_j = 0$. Our strategy is to define linearly independent elements Θ and Φ in $\mathrm{EHom}_{\mathcal{H}_n}(S^\mu, S^\lambda)$ and use Lemmas 2.2 and Lemma 2.3 to show that $\Theta(m_\mu h_{d,t}) = \Phi(m_\mu h_{d,t}) = 0$ for all $1 \leq d \leq 4$ and $1 \leq t \leq \mu_{d+1}$. Theorem 1.2 then follows by Lemma 2.1.

Lemma 2.9. *Suppose that $\mathsf{T} \in \mathcal{T}_0(\lambda, \mu)$ has the form*

$$\mathsf{T} = \begin{array}{l} 1^{ae-3}\, 2^{e-1}\, 3^{\mathsf{T}^3_1}\, 4^{\mathsf{T}^4_1}\, 5^{\mathsf{T}^5_1} \\ 2^{(b-1)e-2}\, 3^{\mathsf{T}^3_2}\, 4^{\mathsf{T}^4_2}\, 5^{\mathsf{T}^5_2} \\ 3^{\mathsf{T}^3_3}\, 4^{\mathsf{T}^4_3}\, 5^{\mathsf{T}^5_3} \end{array}.$$

Then the following results hold.

(1) *Suppose $1 \leq t \leq e-1$. Write $\mathsf{T} \xrightarrow{4,t} \mathsf{S}$ if S is a row-standard ν-tableau formed from T by changing t entries equal to 5 in T into 4s. If $\mathsf{T} \xrightarrow{4,t} \mathsf{S}$ then S is semistandard and*

$$\Theta_{\mathsf{T}}(m_\mu h_{4,t}) = \sum_{\mathsf{T} \xrightarrow{4,t} \mathsf{S}} q^{(\mathsf{T}^4_3+\mathsf{T}^4_2)(\mathsf{S}^4_1-\mathsf{T}^4_1)} \begin{bmatrix} \mathsf{S}^4_1 \\ \mathsf{T}^4_1 \end{bmatrix} q^{\mathsf{T}^4_3(\mathsf{S}^4_2-\mathsf{T}^4_2)} \begin{bmatrix} \mathsf{S}^4_2 \\ \mathsf{T}^4_2 \end{bmatrix} \begin{bmatrix} \mathsf{S}^4_3 \\ \mathsf{T}^4_3 \end{bmatrix} \Theta_{\mathsf{S}}(m_\nu).$$

(2) *Suppose $1 \le t \le e-1$. Write $\mathsf{T} \xrightarrow{3,t} \mathsf{S}$ if S is a row-standard ν-tableau formed from T by changing t entries equal to 4 in T into 3s. If $\mathsf{T} \xrightarrow{3,t} \mathsf{S}$ then S is semistandard and*

$$\Theta_{\mathsf{T}}(m_\mu h_{3,t}) = \sum_{\mathsf{T}\xrightarrow{3,t}\mathsf{S}} q^{(\mathsf{T}_2^3+\mathsf{T}_3^3)(\mathsf{S}_1^3-\mathsf{T}_1^3)} \begin{bmatrix}\mathsf{S}_1^3\\ \mathsf{T}_1^3\end{bmatrix} q^{\mathsf{T}_3^3(\mathsf{S}_2^3-\mathsf{T}_2^3)} \begin{bmatrix}\mathsf{S}_2^3\\ \mathsf{T}_2^3\end{bmatrix}\begin{bmatrix}\mathsf{S}_3^3\\ \mathsf{T}_3^3\end{bmatrix}\Theta_{\mathsf{S}}(m_\nu).$$

(3) *Suppose $1 \le t \le \mu_3 - 1$. Write $\mathsf{T} \xrightarrow{2,t} \mathsf{S}$ if S is a row-standard ν-tableau formed from T by first changing t entries equal to 3 in T into 2s in the second and third rows and then exchanging all entries equal to 2 in row 3 with entries not equal to 2 in row 2. If $\mathsf{T} \xrightarrow{2,t} \mathsf{S}$ then S is semistandard and*

$$\Theta_{\mathsf{T}}(m_\mu h_{2,t}) = \sum_{\mathsf{T}\xrightarrow{2,t}\mathsf{S}} (-1)^{\mathsf{T}_3^3-\mathsf{S}_3^3} q^{\binom{\mathsf{T}_3^3-\mathsf{S}_3^3}{2}+\mathsf{S}_3^3 t}\begin{bmatrix}(b-1)e-2+t-\mathsf{T}_3^3\\ (b-1)e-2-\mathsf{S}_3^3\end{bmatrix} q^{\mathsf{T}_3^4(\mathsf{S}_3^5-\mathsf{T}_3^5)}\begin{bmatrix}\mathsf{S}_3^4\\ \mathsf{T}_3^4\end{bmatrix}\begin{bmatrix}\mathsf{S}_3^5\\ \mathsf{T}_3^5\end{bmatrix}\Theta_{\mathsf{S}}(m_\nu).$$

In particular, $\Theta_{\mathsf{T}}(m_\mu h_{2,t}) = 0$ for $t > e-1$.

(4) *Suppose $1 \le t \le \mu_2 - 1$. Write $\mathsf{T} \xrightarrow{1,t} \mathsf{S}$ if S is a row-standard ν-tableau formed from T by first changing t entries equal to 2 in T into 1s and then exchanging all entries equal to 1 in row 2 with entries not equal to 1 in row 1. If $\mathsf{T} \xrightarrow{1,t} \mathsf{S}$ then S is semistandard and*

$$\Theta_{\mathsf{T}}(m_\mu h_{1,t}) = \sum_{\mathsf{T}\xrightarrow{1,t}\mathsf{S}} (-1)^{\mathsf{T}_2^2-\mathsf{S}_2^2} q^{\binom{\mathsf{T}_2^2-\mathsf{S}_2^2}{2}+\mathsf{S}_2^2 t}\begin{bmatrix}(a-b+1)e-1+t\\ ae-3-\mathsf{S}_2^2\end{bmatrix}$$
$$q^{\mathsf{T}_2^3(\mathsf{S}_2^4-\mathsf{T}_2^4)} q^{(\mathsf{T}_2^3+\mathsf{T}_2^4)(\mathsf{S}_2^5-\mathsf{T}_2^5)}\begin{bmatrix}\mathsf{S}_2^3\\ \mathsf{T}_2^3\end{bmatrix}\begin{bmatrix}\mathsf{S}_2^4\\ \mathsf{T}_2^4\end{bmatrix}\begin{bmatrix}\mathsf{S}_2^5\\ \mathsf{T}_2^5\end{bmatrix}\Theta_{\mathsf{S}}(m_\nu)$$

where $\mathsf{T}_2^2 = (b-1)e-2$. In particular, $\Theta_{\mathsf{T}}(m_\mu h_{1,t}) = 0$ for $t > 2e-2$.

Proof. To check the tableaux S are semistandard, observe that $ae-3 \ge be-3$ and that $(b-1)e-2 \ge ce-3$. Parts (1) and (2) are then just restatements of Lemma 2.2. Now consider (3). Use Lemma 2.2 to write Θ_{T} as a linear combination of terms $\Theta_{\mathsf{R}}(m_\nu)$ where R is formed from T by changing entries equal to 3 into 2s. If $s > 0$ entries are changed in the first row then the term occurs with coefficient a multiple of $\left[\begin{smallmatrix}e-1+s\\ s\end{smallmatrix}\right] = 0$ by Lemma 2.7 so we may assume all entries changed are in the last two rows. It then follows from Lemma 2.3 that $\Theta_{\mathsf{T}}(m_\mu h_{2,t}) = \sum_{\mathsf{T}\xrightarrow{2,t}\mathsf{S}} b(\mathsf{S})\Theta_{\mathsf{S}}(m_\nu)$ where

$$b(\mathsf{S}) = \sum_{j\ge 0}(-1)^j q^{-\binom{j+1}{2}} q^{j(j-\mathsf{T}_3^3+\mathsf{S}_3^3)}\begin{bmatrix}(b-1)e-2+t-j\\ t-j\end{bmatrix} q^{\mathsf{T}_3^3(\mathsf{S}_3^4-\mathsf{T}_3^4)} q^{(\mathsf{T}_3^3+\mathsf{T}_4^4)(\mathsf{S}_3^5-\mathsf{T}_3^5)}\begin{bmatrix}S_3^3\\ T_3^3-j\end{bmatrix}\begin{bmatrix}S_3^4\\ \mathsf{T}_3^4\end{bmatrix}\begin{bmatrix}\mathsf{S}_3^5\\ \mathsf{T}_3^5\end{bmatrix}.$$

Changing the limits of the sum and applying Lemma 2.5 we obtain

$$\begin{aligned}
b(\mathsf{S}) &= (-1)^{\mathsf{T}_3^3-\mathsf{S}_3^3} q^{\mathsf{T}_3^3(\mathsf{S}_3^4-\mathsf{T}_3^4)} q^{(\mathsf{T}_3^3+\mathsf{T}_4^4)(\mathsf{S}_3^5-\mathsf{T}_3^5)} q^{-\binom{\mathsf{T}_3^3-\mathsf{S}_3^3+1}{2}}\begin{bmatrix}S_3^4\\ \mathsf{T}_3^4\end{bmatrix}\begin{bmatrix}\mathsf{S}_3^5\\ \mathsf{T}_3^5\end{bmatrix}\\
&\qquad \sum_{j\ge 0}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}\mathsf{S}_3^3\\ j\end{bmatrix}\begin{bmatrix}(b-1)e-2+t-j-\mathsf{T}_3^3+\mathsf{S}_3^3\\ (b-1)e-2\end{bmatrix}\\
&= (-1)^{\mathsf{T}_3^3-\mathsf{S}_3^3} q^{\mathsf{T}_3^3(\mathsf{S}_3^4-\mathsf{T}_3^4)} q^{(\mathsf{T}_3^3+\mathsf{T}_4^4)(\mathsf{S}_3^5-\mathsf{T}_3^5)} q^{-\binom{\mathsf{T}_3^3-\mathsf{S}_3^3+1}{2}}\begin{bmatrix}S_3^4\\ \mathsf{T}_3^4\end{bmatrix}\begin{bmatrix}\mathsf{S}_3^5\\ \mathsf{T}_3^5\end{bmatrix} q^{\mathsf{S}_3^3(t-\mathsf{T}_3^3+\mathsf{S}_3^3)}\begin{bmatrix}(b-1)e-2+t-\mathsf{T}_3^3\\ (b-1)e-2-\mathsf{S}_3^3\end{bmatrix}\\
&= (-1)^{\mathsf{T}_3^3-\mathsf{S}_3^3} q^{\binom{\mathsf{T}_3^3-\mathsf{S}_3^3}{2}+\mathsf{S}_3^3 t}\begin{bmatrix}(b-1)e-2+t-\mathsf{T}_3^3\\ (b-1)e-2-\mathsf{S}_3^3\end{bmatrix} q^{\mathsf{T}_3^4(\mathsf{S}_3^5-\mathsf{T}_3^5)}\begin{bmatrix}\mathsf{S}_3^4\\ \mathsf{T}_3^4\end{bmatrix}\begin{bmatrix}\mathsf{S}_3^5\\ \mathsf{T}_3^5\end{bmatrix}
\end{aligned}$$

as required.

The proof of part (4) of the lemma follows on identical lines. $\square$

Proposition 2.10. *Define a tableau* $\mathsf{T} \in \mathcal{T}_0(\lambda, \mu)$ *by*

$$\mathsf{T} = \begin{array}{l} 1^{ae-3}\, 2^{e-1}\, 3^{e-1} \\ 2^{(b-1)e-2}\, 3^{e-1} \\ 3^{(c-2)e-1}\, 4^{e-1}\, 5^{e-1} \end{array}$$

and let $\Theta = \Theta_{\mathsf{T}}$. *Then* $\Theta \in \Psi(\mu, \lambda)$.

Proof. Note that T has the form described in Lemma 2.9. Suppose $1 \le t \le e-1$. Then applying Lemma 2.9 and Lemma 2.7,

$$\Theta(m_\mu h_{4,t}) = \begin{bmatrix} e-1+t \\ t \end{bmatrix} \begin{array}{l} 1^{ae-3}\, 2^{e-1}\, 3^{e-1} \\ 2^{(b-1)e-2}\, 3^{e-1} \\ 3^{(c-2)e-1}\, 4^{e-1+t}\, 5^{e-1-t} \end{array} = 0;$$

$$\Theta(m_\mu h_{3,t}) = \begin{bmatrix} (c-2)e-1+t \\ t \end{bmatrix} \begin{array}{l} 1^{ae-3}\, 2^{e-1}\, 3^{e-1} \\ 2^{(b-1)e-2}\, 3^{e-1} \\ 3^{(c-2)e-1+t}\, 4^{e-1-t}\, 5^{e-1} \end{array} = 0;$$

$$\Theta(m_\mu h_{2,t}) = q^{((c-2)e-1)t} \begin{bmatrix} (b-c+1)e-1+t \\ t \end{bmatrix} \begin{array}{l} 1^{ae-3}\, 2^{e-1}\, 3^{e-1} \\ 2^{(b-1)e-2+t}\, 3^{e-1-t} \\ 3^{(c-2)e-1}\, 4^{e-1}\, 5^{e-1} \end{array} = 0.$$

Now suppose $1 \le t \le 2e-2$. Then

$$\Theta(m_\mu h_{1,t}) = \sum_{\mathsf{T} \xrightarrow{1,t} \mathsf{S}} (-1)^{\mathsf{T}_2^2 - \mathsf{S}_2^2} q^{\binom{\mathsf{T}_2^2 - \mathsf{S}_2^2}{2} + \mathsf{S}_2^2 t} \begin{bmatrix} (a-b+1)e-1+t \\ ae-3-\mathsf{S}_2^2 \end{bmatrix} \begin{bmatrix} \mathsf{S}_2^3 \\ e-1 \end{bmatrix}.$$

But if $\mathsf{T} \xrightarrow{d,t} \mathsf{S}$ then $\left[\begin{smallmatrix} \mathsf{S}_2^3 \\ e-1 \end{smallmatrix}\right] = 0$ unless $\mathsf{S}_2^3 = e-1$; and if $\mathsf{S}_2^3 = e-1$ then $1 \le t \le e-1$ and then

$$\begin{bmatrix} (a-b+1)e-1+t \\ ae-3-\mathsf{S}_2^2 \end{bmatrix} = \begin{bmatrix} (a-b+1)e-1+t \\ t \end{bmatrix} = 0$$

by Lemma 2.7. Hence $\Theta(m_\mu h_{d,t}) = 0$ for all $1 \le d \le 4$ and all $1 \le t \le \mu_{d+1}$ as required. □

Proposition 2.11. *Let* $\mathcal{A}$ *denote the set of* λ*-tableaux* A *of type* μ *which have the form*

$$\begin{array}{l} 1^{ae-3}\, 2^{e-1}\, 3^{\mathsf{A}_1^3}\, 4^{\mathsf{A}_1^4}\, 5^{\mathsf{A}_1^5} \\ 2^{(b-1)e-2}\, 3^{\mathsf{A}_2^3}\, 4^{\mathsf{A}_2^4}\, 5^{\mathsf{A}_2^5} \\ 3^{(c-1)e-2}\, 4^{\mathsf{A}_3^4}\, 5^{\mathsf{A}_3^5} \end{array}$$

and $\mathcal{B}$ *denote the set of* λ*-tableaux* B *of type* μ *which have the form*

$$\begin{array}{l} 1^{ae-3}\, 2^{e-1}\, 3^{\mathsf{B}_1^3}\, 4^{\mathsf{B}_1^4}\, 5^{\mathsf{B}_1^5} \\ 2^{(b-1)e-2}\, 3^{\mathsf{B}_2^3}\, 4^{\mathsf{B}_2^4}\, 5^{\mathsf{B}_2^5} \\ 3^{(c-1)e-1}\, 4^{\mathsf{B}_3^4}\, 5^{\mathsf{B}_3^5} \end{array}$$

so that all $\mathsf{A} \in \mathcal{A} \cup \mathcal{B}$ *are semistandard. Set*

$$\Phi = \sum_{\mathsf{A} \in \mathcal{A}} \Theta_{\mathsf{A}} - q \sum_{\mathsf{B} \in \mathcal{B}} \Theta_{\mathsf{B}}.$$

Then $\Phi \in \Psi(\mu, \lambda)$.

Proof. Note that all tableaux $\mathsf{A} \in \mathcal{A} \cup \mathcal{B}$ have the form described in Lemma 2.9 and use the notation of that lemma. For $1 \le d \le 4$ and $1 \le t \le \mu_{d+1}$, let

$$\mathcal{D}(d,t) = \{\mathsf{S} \in \mathcal{T}_0(\lambda, \nu) \mid \mathsf{A} \xrightarrow{d,t} \mathsf{S} \text{ for some } \mathsf{A} \in \mathcal{A} \cup \mathcal{B}\}.$$

For $\mathsf{S} \in \mathcal{D}(d,t)$ define $b_{\mathcal{A}}(\mathsf{S})$ to be the coefficient of $\Theta_{\mathsf{S}}(m_\nu)$ in $\sum_{\mathsf{A} \in \mathcal{A}} \Theta_{\mathsf{A}}(m_\mu h_{d,t})$, define $b_{\mathcal{B}}(\mathsf{S})$ to be its coefficient in $\sum_{\mathsf{B} \in \mathcal{A}} \Theta_{\mathsf{A}}(m_\mu h_{d,t})$ and set $b(\mathsf{S}) = b_{\mathcal{A}}(\mathsf{S}) - q b_{\mathcal{B}}(\mathsf{S})$ to be its coefficient in $\Phi(m_\mu h_{d,t})$.

Take $d=4$ and $1 \le t \le e-1$ and suppose that $\mathsf{S} \in \mathcal{D}(4,t)$. Using Lemma 2.9 and applying Lemma 2.8 and Lemma 2.7 we have

$$\begin{aligned}
b_{\mathcal{A}}(\mathsf{S}) &= \sum_{\substack{\mathsf{A}\in\mathcal{A}\\ \mathsf{A}\xrightarrow{d,t}\mathsf{S}}} q^{(\mathsf{A}^4_3+\mathsf{A}^4_2)(\mathsf{S}^4_1-\mathsf{A}^4_1)} \begin{bmatrix}\mathsf{S}^4_1\\ \mathsf{A}^4_1\end{bmatrix} q^{\mathsf{A}^4_3(\mathsf{S}^4_2-\mathsf{A}^4_2)} \begin{bmatrix}\mathsf{S}^4_2\\ \mathsf{A}^4_2\end{bmatrix}\begin{bmatrix}\mathsf{S}^4_3\\ \mathsf{A}^4_3\end{bmatrix}\\
&= \sum_{\mathsf{A}^4_1+\mathsf{A}^4_2+\mathsf{A}^4_3=e-1} q^{(\mathsf{A}^4_3+\mathsf{A}^4_2)(\mathsf{S}^4_1-\mathsf{A}^4_1)} \begin{bmatrix}\mathsf{S}^4_1\\ \mathsf{A}^4_1\end{bmatrix} q^{\mathsf{A}^4_3(\mathsf{S}^4_2-\mathsf{A}^4_2)} \begin{bmatrix}\mathsf{S}^4_2\\ \mathsf{A}^4_2\end{bmatrix}\begin{bmatrix}\mathsf{S}^4_3\\ \mathsf{A}^4_3\end{bmatrix}\\
&= \begin{bmatrix}\mathsf{S}^4_1+\mathsf{S}^4_2+\mathsf{S}^4_3\\ \mathsf{A}^4_1+\mathsf{A}^4_2+\mathsf{A}^4_3\end{bmatrix}\\
&= \begin{bmatrix}e-1+t\\ t\end{bmatrix}\\
&= 0.
\end{aligned}$$

An identical argument shows that $b_{\mathcal{B}}(\mathsf{S})$ is also zero.

Now take $d=3$ and $1 \le t \le e-1$. Suppose that $\mathsf{S} \in \mathcal{D}(d,t)$. Then

$$\begin{aligned}
b(\mathsf{S}) = &\sum_{\substack{\mathsf{A}\in\mathcal{A}\\ \mathsf{A}\xrightarrow{d,t}\mathsf{S}}} q^{(\mathsf{A}^3_2+\mathsf{A}^3_3)(\mathsf{S}^3_1-\mathsf{A}^3_1)} q^{A^3_3(\mathsf{S}^3_2-A^3_2)} \begin{bmatrix}\mathsf{S}^3_1\\ \mathsf{A}^3_1\end{bmatrix}\begin{bmatrix}\mathsf{S}^3_2\\ \mathsf{A}^3_2\end{bmatrix}\begin{bmatrix}\mathsf{S}^3_3\\ \mathsf{A}^3_3\end{bmatrix}\\
&- q\sum_{\substack{\mathsf{B}\in\mathcal{B}\\ \mathsf{B}\xrightarrow{d,t}\mathsf{S}}} q^{(\mathsf{B}^3_2+\mathsf{B}^3_3)(\mathsf{S}^3_1-\mathsf{B}^3_1)} q^{B^3_3(\mathsf{S}^3_2-B^3_2)} \begin{bmatrix}\mathsf{S}^3_1\\ \mathsf{B}^3_1\end{bmatrix}\begin{bmatrix}\mathsf{S}^3_2\\ \mathsf{B}^3_2\end{bmatrix}\begin{bmatrix}\mathsf{S}^3_3\\ \mathsf{B}^3_3\end{bmatrix}\\
= &\, q^{((c-1)e-2)((c-1)e-2-\mathsf{S}^3_3+t)} \begin{bmatrix}S^3_3\\ (c-1)e-2\end{bmatrix} \sum_{A^3_1+\mathsf{A}^3_2=e-1} q^{\mathsf{A}^3_2(\mathsf{S}^3_1-\mathsf{A}^3_1)} \begin{bmatrix}\mathsf{S}^3_2\\ \mathsf{A}^3_2\end{bmatrix}\begin{bmatrix}\mathsf{S}^3_1\\ \mathsf{A}^3_1\end{bmatrix}\\
&- q^{((c-1)e-1)((c-1)e-1-\mathsf{S}^3_3+t)+1} \begin{bmatrix}S^3_3\\ (c-1)e-1\end{bmatrix} \sum_{B^3_1+\mathsf{B}^3_2=e-2} q^{\mathsf{B}^3_2(\mathsf{S}^3_1-\mathsf{B}^3_1)} \begin{bmatrix}\mathsf{S}^3_2\\ \mathsf{B}^3_2\end{bmatrix}\begin{bmatrix}\mathsf{S}^3_1\\ \mathsf{B}^3_1\end{bmatrix}\\
= &\, q^{((c-1)e-2)((c-1)e-2-\mathsf{S}^3_3+t)} \begin{bmatrix}S^3_3\\ (c-1)e-2\end{bmatrix}\begin{bmatrix}\mathsf{S}^3_2+\mathsf{S}^3_1\\ e-1\end{bmatrix} - q^{((c-1)e-1)((c-1)e-1-\mathsf{S}^3_3+t)+1} \begin{bmatrix}S^3_3\\ (c-1)e-1\end{bmatrix}\begin{bmatrix}\mathsf{S}^3_2+\mathsf{S}^3_1\\ e-2\end{bmatrix}
\end{aligned}$$

where, by Lemma 2.6, $\left[{S^3_3 \atop (c-1)e-2}\right]$ and $\left[{S^3_3 \atop (c-1)e-1}\right]$ are both zero unless $\mathsf{S}^3_3 = (c-1)e-2$ or $\mathsf{S}^3_3 = (c-1)e-1$. If $\mathsf{S}^3_3 = (c-1)e-2$ then $\mathsf{S}^3_1 + S^3_2 = e-1+t$ and

$$b(\mathsf{S}) = q^{((c-1)e-1)t}\begin{bmatrix}e-1+t\\ t\end{bmatrix} = 0$$

by Lemma 2.7. If $\mathsf{S}^3_3 = (c-1)e-1$ then $\mathsf{S}^3_1 + S^3_2 = e-2+t$. Note that $[(c-1)e-1] = -q^{(c-1)e-1}$. Applying Lemma 2.4 and Lemma 2.7 we have

$$\begin{aligned}
b(\mathsf{S}) &= q^{((c-1)e-2)(t-1)}[(c-1)e-1]\begin{bmatrix}e-2+t\\ e-1\end{bmatrix} - q^{((c-1)e-1)t+1}\begin{bmatrix}e-2+t\\ e-2\end{bmatrix}\\
&= -q^{((c-1)e-2)t+1}\left(\begin{bmatrix}e-2+t\\ e-1\end{bmatrix} + q^t\begin{bmatrix}e-2+t\\ e-2\end{bmatrix}\right)\\
&= -q^{((c-1)e-2)t+1}\begin{bmatrix}e-1+t\\ t\end{bmatrix}\\
&= 0
\end{aligned}$$

as required.

Now take $d = 2$ and $1 \le t \le e-1$ and suppose that $\mathsf{S} \in \mathcal{D}(2,t)$. If $\mathsf{A} \in \mathcal{A}$ note that $\mathsf{A}_3^3 = (c-1)e-2$. Then

$$\begin{aligned}
b_{\mathcal{A}}(\mathsf{S}) &= \sum_{\mathsf{A} \xrightarrow{2,t} \mathsf{S}} (-1)^{\mathsf{A}_3^3-\mathsf{S}_3^3} q^{\binom{\mathsf{A}_3^3-\mathsf{S}_3^3}{2}+\mathsf{S}_3^3 t} \begin{bmatrix}(b-1)e-2+t-\mathsf{A}_3^3\\(b-1)e-2-\mathsf{S}_3^3\end{bmatrix} q^{\mathsf{A}_3^4(\mathsf{S}_3^5-\mathsf{A}_4^5)} \begin{bmatrix}\mathsf{S}_3^4\\\mathsf{A}_3^4\end{bmatrix}\begin{bmatrix}\mathsf{S}_3^5\\\mathsf{A}_3^5\end{bmatrix}\\
&= (-1)^{(c-1)e-2-\mathsf{S}_3^3} q^{\binom{(c-1)e-2-\mathsf{S}_3^3}{2}+\mathsf{S}_3^3 t} \begin{bmatrix}(b-c)e+t\\(b-1)e-2-\mathsf{S}_3^3\end{bmatrix} \sum_{\mathsf{A}_3^4+\mathsf{A}_3^5=e-1} q^{\mathsf{A}_3^4(\mathsf{S}_3^5-\mathsf{A}_4^5)} \begin{bmatrix}\mathsf{S}_3^4\\\mathsf{A}_3^4\end{bmatrix}\begin{bmatrix}\mathsf{S}_3^5\\\mathsf{A}_3^5\end{bmatrix}\\
&= (-1)^{(c-1)e-2-\mathsf{S}_3^3} q^{\binom{(c-1)e-2-\mathsf{S}_3^3}{2}+\mathsf{S}_3^3 t} \begin{bmatrix}(b-c)e+t\\(b-1)e-2-\mathsf{S}_3^3\end{bmatrix}\begin{bmatrix}S_3^4+\mathsf{S}_3^5\\e-1\end{bmatrix}
\end{aligned}$$

and the same argument shows that

$$b_{\mathcal{B}}(\mathsf{S}) = (-1)^{(c-1)e-1-\mathsf{S}_3^3} q^{\binom{(c-1)e-1-\mathsf{S}_3^3}{2}+\mathsf{S}_3^3 t} \begin{bmatrix}(b-c)e-1+t\\(b-1)e-2-\mathsf{S}_3^3\end{bmatrix}\begin{bmatrix}S_3^4+\mathsf{S}_3^5\\e-2\end{bmatrix}.$$

Note that if $\mathsf{A} \xrightarrow{2,t} \mathsf{S}$ for some $\mathsf{A} \in \mathcal{A}$ then $e-1 \le \mathsf{S}_3^4+\mathsf{S}_3^5 \le 2e-2$ and if $\mathsf{B} \xrightarrow{2,t} \mathsf{S}$ for some $\mathsf{B} \in \mathcal{B}$ then $e-2 \le \mathsf{S}_3^4+\mathsf{S}_3^5 \le 2e-3$. So by Lemma 2.7, $b(\mathsf{S}) = 0$ unless $\mathsf{S}_3^4+\mathsf{S}_3^5 = e-2$ or $\mathsf{S}_3^4+\mathsf{S}_3^5 = e-1$. If $\mathsf{S}_3^4+\mathsf{S}_3^5 = e-2$ then

$$b(\mathsf{S}) = (-q)q^{\mathsf{S}_3^3 t}\begin{bmatrix}(b-c)e-1+t\\t\end{bmatrix} = 0$$

by Lemma 2.7. If $\mathsf{S}_3^4+\mathsf{S}_3^5 = e-1$ then recall that $[e-1] = -q^{e-1} = -q^{(b-c)e-1}$. Then

$$\begin{aligned}
b(\mathsf{S}) &= q^{\mathsf{S}_3^3 t}\begin{bmatrix}(b-c)e+t\\t\end{bmatrix} + (q)q^{\mathsf{S}_3^3 t}\begin{bmatrix}(b-c)e-1+t\\t-1\end{bmatrix}[e-1]\\
&= q^{\mathsf{S}_3^3 t}\left(\begin{bmatrix}(b-c)e+t\\t\end{bmatrix} - q^{(b-c)e}\begin{bmatrix}(b-1)e+t-1\\t-1\end{bmatrix}\right)\\
&= q^{\mathsf{S}_3^3 t}\begin{bmatrix}(b-c)e+t-1\\t\end{bmatrix}\\
&= 0
\end{aligned}$$

by Lemma 2.4 and Lemma 2.7.

Finally take $d = 1$ and $1 \le t \le 2e-2$ and suppose that $\mathsf{S} \in \mathcal{D}(1,t)$. By Lemma 2.9

$$\begin{aligned}
b_{\mathcal{A}}(\mathsf{S}) &= \sum_{\mathsf{A} \xrightarrow{2,t} \mathsf{S}} (-1)^{\mathsf{A}_2^2-\mathsf{S}_2^2} q^{\binom{\mathsf{A}_2^2-\mathsf{S}_2^2}{2}+\mathsf{S}_2^2 t} \begin{bmatrix}(a-b+1)e-1+t\\ae-3-\mathsf{S}_2^2\end{bmatrix} q^{\mathsf{A}_2^3(\mathsf{S}_2^4-\mathsf{A}_2^4)} q^{(\mathsf{A}_2^3+\mathsf{A}_2^4)(\mathsf{S}_2^5-\mathsf{A}_2^5)} \begin{bmatrix}\mathsf{S}_2^3\\\mathsf{A}_2^3\end{bmatrix}\begin{bmatrix}\mathsf{S}_2^4\\\mathsf{A}_2^4\end{bmatrix}\begin{bmatrix}\mathsf{S}_2^5\\\mathsf{A}_2^5\end{bmatrix}\\
&= (-1)^{\mathsf{A}_2^2-\mathsf{S}_2^2} q^{\binom{\mathsf{A}_2^2-\mathsf{S}_2^2}{2}+\mathsf{S}_2^2 t} \begin{bmatrix}(a-b+1)e-1+t\\ae-3-\mathsf{S}_2^2\end{bmatrix} \sum_{\mathsf{A}_2^3+\mathsf{A}_2^4+\mathsf{A}_2^5=e-1} q^{\mathsf{A}_2^3(\mathsf{S}_2^4-\mathsf{A}_2^4)} q^{(\mathsf{A}_2^3+\mathsf{A}_2^4)(\mathsf{S}_2^5-\mathsf{A}_2^5)} \begin{bmatrix}\mathsf{S}_2^3\\\mathsf{A}_2^3\end{bmatrix}\begin{bmatrix}\mathsf{S}_2^4\\\mathsf{A}_2^4\end{bmatrix}\begin{bmatrix}\mathsf{S}_2^5\\\mathsf{A}_2^5\end{bmatrix}\\
&= (-1)^{\mathsf{A}_2^2-\mathsf{S}_2^2} q^{\binom{\mathsf{A}_2^2-\mathsf{S}_2^2}{2}+\mathsf{S}_2^2 t} \begin{bmatrix}(a-b+1)e-1+t\\ae-3-\mathsf{S}_2^2\end{bmatrix}\begin{bmatrix}\mathsf{S}_2^3+\mathsf{S}_2^4+\mathsf{S}_2^5\\e-1\end{bmatrix}.
\end{aligned}$$

Since $e-1 \le \mathsf{S}_2^3+\mathsf{S}_2^4+\mathsf{S}_2^5 \le 2e-2$, Lemma 2.7 shows that the last term is zero unless $\mathsf{S}_2^3+\mathsf{S}_2^4+\mathsf{S}_2^5 = e-1$. In this case $1 \le t \le e-1$ and $\mathsf{S}_2^2 = (b-1)e-2$ and so $b_{\mathcal{A}}(\mathsf{S})$ has a factor

$$\begin{bmatrix}(a-b+1)e-1+t\\t\end{bmatrix} = 0$$

by Lemma 2.7. An identical argument shows that $b_{\mathcal{B}}(\mathsf{S}) = 0$.

This completes the proof that $\Phi(m_\mu h_{d,t}) = 0$ for all $1 \le d \le 4$ and all $1 \le t \le \mu_{d+1}$. □

References

[1] *J. Chuang and K.M. Tan*, Filtrations in Rouquier blocks of symmetric groups and Schur algebras, Proc. London Math. Soc. (3), **86** (2003), 685706.

[2] *R. Dipper and G. James*, q-Tensor space and q-Weyl modules, Trans. Amer. Math. Soc., **327** (1991), 251–282.

[3] *C. Dodge*, Large dimensional homomorphism spaces between Specht modules for symmetric groups, J. Pure App. Alg., to appear; arXiv:1103.0246.

[4] *S. Donkin*, Tilting modules for algebraic groups and finite dimensional algebras, A handbook of tilting theory (2007), London Math. Soc. Lect. Notes 332, Cambridge University Press.

[5] *S. Lyle*, On homomorphisms indexed by semistandard tableaux; arXiv:1101.3192.

[6] *S. Lyle and A. Mathas*, Row and column removal theorems for homomorphisms of Specht modules and Weyl modules, J. Alg. Comb. **22** (2005), 151–179.

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